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Solvability Criteria for Certain N-Dimensional Moment Problems*

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1. Introduction

Let K be a closed set in the real N-dimensional Euclidean space E_N. To formulate the moment problem for K the following notation is convenient. A generic point in E_N is an N-tuple $\xi = (\xi_1, ..., \xi_N)$ of real numbers, and a multi-index α is an N-tuple $\alpha = (\alpha_1, ..., \alpha_N)$ of non-negative integers. The monomial function ξ^α is defined by

$$\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_N^{\alpha_N}.$$

If for every multi-index α there is given a real number $\lambda(\alpha)$ then the multi-indexed sequence $\{\lambda(\alpha)\}$ is called a candidate moment sequence. In the K-moment problem it is required to find necessary and sufficient conditions, expressed in terms of the numbers $\lambda(\alpha)$, in order that the candidate moment sequence $\{\lambda(\alpha)\}$ have a representation

$$\lambda(\alpha) = \int_K \xi^\alpha \, d\mu, \tag{1}$$

where μ is a *non-negative* Borel measure with support contained in K. If there is such a measure μ then $\{\lambda(\alpha)\}$ is called a *K-moment sequence* and μ is called a *solution of the K-moment problem.*

For $N = 1$ the moment problem has been studied intensively and an elaborate theory has been developed. For an account of the methods and results of this theory, and its history see [1, 4, 7]. For $N > 1$ the known theory of the moment problem is comparatively meager. Hildebrandt and Schoenberg [3] have necessary and sufficient conditions for solvability in the case when K is an N-dimensional cube, and Stancu [8] for the case of an N-

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dimensional simplex. Results for other specific sets K in E_N, $N > 1$, appear to be unknown. However, for $N \geqslant 1$ there are some general results known concerning the uniqueness of the solution of the moment problem, and for $K = E_N$ sufficient criteria for solvability are known provided the moments satisfy a growth condition which guarantees uniqueness of the solution [2].

The objective here is to provide necessary and sufficient criteria for solvability of the moment problem for the cases in which K is one of the following: a (solid) sphere or its boundary; a polysphere (e.g., polydisc) or its distinguished boundary; a torus; a cylinder or certain parts thereof.

2. Notation

Real or complex valued functions defined on E_N will be denoted by lower-case Latin letters. If f is a function its value at a point $\xi = (\xi_1, ..., \xi_N)$ in E_N will be denoted by $f(\xi)$ or $f(\xi_1, ..., \xi_N)$. The adjoint function f^* is defined by

$$f^*(\xi) = \overline{f(\xi)},$$

where the bar signifies complex conjugation. A function f is real if and only if $f = f^*$. The coordinate functions x_j, $j = 1, ..., N$, are defined by

$$x_j(\xi) = \xi_j.$$

They are real and for any function f

$$f = f(x_1, ..., x_N),$$

where the right side is interpreted as a composite function. The radial function r is defined by

$$r = (x_1^2 + \cdots + x_N^2)^{1/2}.$$

If $\alpha = (\alpha_1, ..., \alpha_N)$ is a multi-index, its *order* is $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and the corresponding monomial

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$$

is of degree $|\alpha|$. If $\alpha = (\alpha_1, ..., \alpha_N)$ and $\beta = (\beta_1, ..., \beta_N)$ then $\alpha + \beta = (\alpha_1 + \beta_1, ..., \alpha_N + \beta_N)$.

The set of all polynomials in the N variables ξ_j with complex coefficients will be denoted by $\mathscr{P}$. With the usual rules for algebraic operations $\mathscr{P}$ is a commutative complex algebra with involution $p \to p^*$, and the constant function 1 is an identity element for $\mathscr{P}$.

3. Moment Functionals

Corresponding to any multi-sequence $\{\lambda(\alpha)\}$ of real or complex numbers there is a linear function L defined on $\mathscr{P}$ as follows. If

$$p = \sum_{\alpha} \pi(\alpha)\, x^{\alpha}$$

is the representation of a polynomial p as a finite linear combination of monomials then

$$L(p) = \sum_{\alpha} \pi(\alpha)\, \lambda(\alpha).$$

Conversely, if L is a given linear functional on $\mathscr{P}$, it is obtained in this way from the multi-sequence $\{\lambda(\alpha)\}$, where

$$\lambda(\alpha) = L(x^{\alpha}).$$

The linear functional L on $\mathscr{P}$ is called a *K-moment functional* if it has a representation

$$L(p) = \int_K p\, d\mu, \tag{2}$$

where μ is a non-negative Borel measure in K. Evidently L is a K-moment functional if and only if the corresponding multi-sequence $\{\lambda(\alpha)\}$ is a K-moment sequence.

Solvability criteria for the K-moment problem can often be most conveniently summarized as conditions to be satisfied by the linear functional L. Such conditions are regarded as effective solvability criteria if they are readily converted into equivalent conditions on the multi-sequence $\{\lambda(\alpha)\}$. For example, the condition

(i) $\qquad L(pp^*) \geqslant 0 \qquad$ for every $\quad p \in \mathscr{P}$,

is obviously a necessary condition for the existence of a representation (2). The equivalent condition on the sequence $\{\lambda(\alpha)\}$ is obtained as follows. If

$$p = \sum_{\alpha} \pi(\alpha)\, x^{\alpha}$$

is a polynomial, then

$$pp^* = \sum_{\alpha} \sum_{\beta} \pi(\alpha)\, \overline{\pi(\beta)}\, x^{\alpha+\beta}$$

and

$$L(pp^*) = \sum_\alpha \sum_\beta \pi(\alpha)\,\overline{\pi(\beta)}\,\lambda(\alpha+\beta). \tag{3}$$

The right member of (3) is a Hermitian form in the variables $\pi(\alpha)$, and condition (i) is the assertion that this form be positive semi-definite. This will be the case if and only if every principal minor of the matrix of the Hermitian form is non-negative. Thus condition (i) is equivalent to the condition

(î) for every finite sequence $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$ of distinct multi-indices $\det \lambda(\alpha^{(j)} + \alpha^{(k)}) \geqslant 0$.

Condition (i) or equivalently (î) is the classical necessary and sufficient condition for solvability of the moment problem in $K = E_1$.

A set B in E_N is called an *algebraic set* if there is a non-zero polynomial q which vanishes everywhere on B. In E_1 the algebraic sets are the finite sets, in E_2 they are the algebraic curves or subsets thereof.

Every E_N-moment functional L satisfies condition (i), and this condition is part of the criteria in some of the theorems below. In order that the representing measure μ of L be not supported by any algebraic set it is clear from (2) that it is necessary and sufficient that (i) be satisfied with strict inequality for every non-zero $p \in \mathscr{P}$.

4. Statement of Main Results

Let S be the unit sphere and Σ the surface:

$$S = \{\xi; r^2(\xi) \leqslant 1\}, \tag{4}$$

$$\Sigma = \{\xi; r^2(\xi) = 1\}. \tag{5}$$

Theorem 1. *In order that a linear functional L on $\mathscr{P}$ be an S-moment functional it is necessary and sufficient that L satisfy the conditions* (i) $L(pp^*) \geqslant 0$ *for every* $p \in \mathscr{P}$, *and* (ii) $L((1-r^2)pp^*) \geqslant 0$ *for every* $p \in \mathscr{P}$.

Criteria for solvability of the Σ-moment problem can be deduced from Theorem 1 in various forms. When $N = 2$, Σ is a circle and the corresponding moment problem has been intensively studied as the trigonometric moment problem. It therefore seems most natural to formulate the Σ-moment problem as a harmonic moment problem.

A not necessarily homogeneous polynomial h in $\mathscr{P}$ is called a harmonic polynomial if it satisfies the Laplace equation in E_N. It is well known that for every polynomial p there is a unique harmonic polynomial h such that

$$p = h + (1 - r^2)q \qquad q \in \mathscr{P}.$$

The values of p and h on the boundary Σ of S are the same, and h is the unique solution of the Laplace equation in S with these boundary values.

A complex valued function $\hat{h}$ defined on Σ is called a *spherical harmonic* if there is a harmonic polynomial h in $\mathscr{P}$ such that $h = \hat{h}$ on Σ. Under algebraic operations defined pointwise on Σ, the collection of all spherical harmonics forms a complex commutative algebra $\mathscr{H}$ with an involution $\hat{h} \to \hat{h}^*$ and an identity, the constant function 1. The product of two elements $\hat{h}_1$, $\hat{h}_2$ of $\mathscr{H}$ will be denoted by $\hat{h}_1 \otimes \hat{h}_2$. To see that this pointwise product is indeed a spherical harmonic, reason as follows. Let h_1 and h_2 be the extensions of $\hat{h}_1$ and $\hat{h}_2$ to harmonic polynomials and form the polynomial $k = h_1 h_2$. Then k agrees with $\hat{h}_1 \otimes \hat{h}_2$ on Σ, and there is a unique harmonic polynomial h_3 such that $k = h_3 + (1 - r^2)q$, $q \in \mathscr{P}$. The polynomial h_3 is a harmonic extension of $\hat{h}_1 \otimes \hat{h}_2$, which latter is therefore a spherical harmonic.

Another equivalent view of the algebra $\mathscr{H}$ of spherical harmonics is the following. The harmonic polynomials form a complete system of representatives for the cosets in $\mathscr{P}$ of the principal ideal generated by $1 - r^2$, and so the space of harmonic polynomials is naturally endowed with the algebraic operations of the quotient algebra $\mathscr{P}/(1 - r^2)$. This algebra is isomorphic to the algebra $\mathscr{H}$, the isomorphism mediated by restriction of functions to Σ.

A subset B of Σ is called *harmonically algebraic* if there is a non-zero spherical harmonic which vanishes at every point of B. A linear functional M on the algebra $\mathscr{H}$ of spherical harmonics is called a *harmonic moment functional* if there is a non-negative Borel measure μ on Σ such that

$$M(\hat{h}) = \int_\Sigma \hat{h}\, d\mu \qquad \text{for every} \quad \hat{h} \in \mathscr{H}. \tag{6}$$

THEOREM 2. *In order that a linear functional M on $\mathscr{H}$ be a harmonic moment functional it is necessary and sufficient that*

$$\text{(iii)} \qquad M(\hat{h} \otimes \hat{h}^*) \geqslant 0 \qquad \text{for every} \quad \hat{h} \in \mathscr{H},$$

and in order that in addition the representing measure μ be not supported by any harmonically algebraic set, it is necessary and sufficient that strict inequality hold in (iii) *for every non-zero $\hat{h} \in \mathscr{H}$.*

Theorem 2 is trivial for $N = 1$, and for $N = 2$ coincides with the well known solvability criteria for the trigonometric moment problem.

Conventionally, a polydisc is a direct profuct of finitely many discs. For example, the subset of E_4 defined by

$$\{\xi; \xi_1^2 + \xi_2^2 \leqslant 1, \xi_3^2 + \xi_4^2 \leqslant 1\}$$

is a bi-disc. The subset of E_{2N} defined by

$$S^2 = \{\xi; \xi_1^2 + \xi_2^2 + \cdots + \xi_N^2 \leqslant 1, \xi_{N+1}^2 + \cdots + \xi_{2N}^2 \leqslant 1\} \tag{7}$$

is a bi-sphere. For $N = 1$ the bi-sphere is a square, for $N = 2$ it is a bi-disc. The two following theorems are stated for the bi-sphere, but both the methods of proof and the statements extend in an utterly straightforward way to direct product of finitely many spheres not necessarily all of the same dimension.

The first N coordinate functions for E_{2N} will be denoted by x_j, $j = 1,..., N$, and the last N by y_j, $j = 1,..., N$. Thus

$$x_j(\xi) = \xi_j, \qquad y_j(\xi) = \xi_{N+j}, \qquad j = 1,..., N.$$

The two partial radial functions s and t are defined by

$$s = (x_1^2 + \cdots + x_N^2)^{1/2}, \qquad t = (y_1^2 + \cdots + y_N^2)^{1/2}$$

so that $s^2 + t^2 = r^2$. The general monomial is of the form $x^\alpha y^\beta$, where $\alpha = (\alpha_1,..., \alpha_N)$ and $\beta = (\beta_1,..., \beta_N)$ are multi-indices for E_N. The algebra of all polynomials in the $2N$ variables will be denoted by $\mathscr{P}^2$.

The Bergman–Silov boundary, or distinguished boundary, of S^2 is the set

$$\Sigma^2 = \{\xi; s^2(\xi) = 1, t^2(\xi) = 1\}. \tag{8}$$

Any monomial $x^\alpha y^\beta$ is equal on Σ^2 to a product

$$x^\alpha y^\beta = h(x_1,..., x_N)\, k(y_1,..., y_N),$$

where h and k are harmonic polynomials. A function $\hat{f}$ defined on Σ^2 is called a *double spherical harmonic* if it is equal on Σ^2 to a sum of products of harmonic polynomials in $x_1,..., x_N$ and harmonic polynomials in $y_1,..., y_N$. Every $p \in \mathscr{P}^2$ is equal on Σ^2 to a unique double spherical harmonic. Under pointwise algebraic operations the set of all double spherical harmonics forms a commutative algebra $\mathscr{H}^2$ with an involution and an identity element. The product of two elements $\hat{f}$, $\hat{g}$ of $\mathscr{H}^2$ will be written $\hat{f} \otimes \hat{g}$.

A linear functional M on $\mathscr{H}^2$ is called a double *harmonic moment* functional if there is a non-negative Borel measure μ on Σ^2 such that

$$L(\hat{f}) = \int_{\Sigma^2} \hat{f}\, d\mu \qquad \text{for every} \quad \hat{f} \in \mathscr{H}^2.$$

A subset B of Σ^2 is called a *double harmonically algebraic set* if there is a non-zero double harmonic polynomial which vanishes everywhere on B.

THEOREM 3. *In order that a linear functional L on* $\mathscr{P}^2$ *be an* S^2*-moment functional it is necessary and sufficient that*

$$\text{(i)} \qquad L(pp^*) \geqslant 0 \qquad \text{for all} \quad p \in \mathscr{P}^2.$$

and

$$\text{(iv)} \qquad \begin{array}{l} L((1-s^2)\,pp^*) \geqslant 0 \\ L((1-t^2)\,pp^*) \geqslant 0 \qquad \text{for all} \quad p \in \mathscr{P}^2. \end{array}$$

THEOREM 4. *In order that a linear functional M on the algebra* $\mathscr{H}^2$ *be a double harmonic moment functional, it is necessary and sufficient that*

$$\text{(v)} \qquad M(\hat{f} \otimes \hat{f}^*) \geqslant 0 \qquad \text{for every} \quad \hat{f} \in \mathscr{H}^2,$$

and in order that in addition the representing measure μ *be not supported by any double harmonically algebraic set, it is necessary and sufficient that strict inequality hold in* (v) *for every non-zero* $\hat{f} \in \mathscr{H}^2$.

When $N = 1$, Theorem 3 is a version of a theorem of Hildebrandt and Schoenberg quoted in the following section, and Theorem 4 is trivial. When $N = 2$, Theorem 4 is concerned with double trigonometric polynomials

$$f = \sum_{k=-n}^{n} \sum_{l=-m}^{m} \delta(k, m)\, e^{ik\theta} e^{il\varphi}.$$

If, in E_3, the disc with center $(1-\tau, 0, 0)$ and radius τ, where $0 < \tau \leqslant 1/2$, be rotated about the x_3-axis, the domain swept out is a (solid) torus T defined in terms of the polynomial

$$f_T = 4(1-\tau)^2\,(r^2 - x_3^2) - (r^2 + 1 - 2\tau)^2 \tag{9}$$

by the inequality

$$T = \{\xi; f_T(\xi) \geqslant 0\}. \tag{10}$$

When $\tau = \frac{1}{2}$ the surface of T has a singularity which is harmless here.

THEOREM 5. *In order that a linear functional L on* $\mathscr{P}$, $(N = 3)$, *be a T-moment functional it is necessesary and sufficient that*

$$\text{(i)} \qquad L(pp^*) \geqslant 0 \qquad \text{for every} \quad p \in \mathscr{P}.$$

and

(vi) $$L(f_T\, pp^*) \geqslant 0 \qquad \textit{for every} \quad p \in \mathscr{P}.$$

In E_{N+1}, with the partial radius

$$r_N = (x_1^2 + \cdots + x_N^2)^{1/2}$$

a spherical cylinder Z is defined by

$$Z = \{\xi;\, r_N^2(\xi) \leqslant 1, -\infty < \xi_{N+1} < \infty\} \tag{11}$$

and its positive half Z^+ by

$$Z^+ = \{\xi;\, \xi \in Z, 0 \leqslant \xi_{N+1} < \infty\}. \tag{12}$$

In this context $\mathscr{P}$ is the algebra of all polynomials in the $N + 1$ variables.

THEOREM 6. *In order that a linear functional L on $\mathscr{P}$ be a Z-moment functional it is necessary and sufficient that*

(i) $$L(pp^*) \geqslant 0 \qquad \textit{for every} \quad p \in \mathscr{P},$$

and

(vii) $$L((1 - r_N^2)\, pp^*) \geqslant 0 \qquad \textit{for every} \quad p \in \mathscr{P}.$$

THEOREM 7. *In order that a linear functional L on $\mathscr{P}$ be a Z^+-moment functional it is necessary and sufficient that* (i) *and* (vii) *of Theorem* 6 *hold and*

(viii) $$L(x_{N+1}\, pp^*) \geqslant 0 \qquad \textit{for every} \quad p \in \mathscr{P}.$$

5. THEOREM OF HILDEBRANDT AND SCHOENBERG

In [3], Hildebrandt and Schoenberg gave a solvability criterion for the moment problem in a cube. They stated their result for the unit cube in the positive orthant. By a translation of axes and change of scale an equivalent result is obtained for the cube

$$C = \{\xi;\, -1 \leqslant \xi_j \leqslant 1, j = 1,..., N\}. \tag{13}$$

If α and β are multi-indices then $(1 - x)^\alpha\, (1 + x)^\beta$ is a hort notation for

$$(1 - x)^\alpha\, (1 + x)^\beta = (1 - x_1)^{\alpha_1} \cdots (1 - x_N)^{\alpha_N}\, (1 + x_1)^{\beta_1} \cdots (1 + x_N)^{\beta_N} \tag{14}$$

Their result is the following.

THEOREM A. *In order that a linear functional L on $\mathscr{P}$ be a C-moment functional it is necessary and sufficient that for every pair of multi-indices α, β*

$$L((1-x)^{\alpha}(1+x)^{\beta}) \geqslant 0. \tag{15}$$

The following theorem [7, p. 11] can be obtained as an easy deduction from the Weierstrass polynomial approximation theorem.

THEOREM B. *If an E_N-moment problem has a solution with compact support, then it has only one solution.*

6. PROOFS

PROPOSITION 1. *If L is an E_N-moment functional with a representing measure μ with compact support K, and f is a polynomial such that $L(fpp^*) \geqslant 0$ for every $p \in \mathscr{P}$, then K is contained in the closed set*

$$F = \{\xi; f(\xi) \geqslant 0\}.$$

Proof. Let B be a compact subset of the complement of F. By the Weierstrass polynomial approximation theorem there is a real polynomial p such that $p > 1$ on B and $|p| \leqslant \frac{1}{2}$ on $K \cap F$. For any integer $n \geqslant 0$

$$0 \leqslant L(fp^{2n}) = \int_{K \cap F} fp^{2n}\, d\mu + \int_{E_N - F} fp^{2n}\, d\mu.$$

The first integral $\to 0$ as $n \to \infty$, the second has a non-positive integrand and

$$\int_{E_N - F} fp^{2n}\, d\mu \leqslant \int_B f\, d\mu$$

which is <0 if $\mu(B) \neq 0$. This would be a contradiction so $\mu(B) = 0$ and $K \subseteq F$.

Proof of Theorem 1

The identities

$$1 - x_1 = \tfrac{1}{2}[(1-x_1)^2 + x_2^2 + \cdots + x_N^2] + (1-r^2)\cdot\tfrac{1}{2}, \tag{16}$$

$$1 + x_1 = \tfrac{1}{2}[(1+x_1)^2 + x_2^2 + \cdots + x_N^2] + (1-r^2)\tfrac{1}{2} \tag{17}$$

show that $1 - x_1$ and $1 + x_1$ belong to the set Q of all polynomials q representable in the form

$$q = a + (1 - r^2)b,$$

where a and b are finite sums of squares of real polynomials. Similar identities obtained by permuting the x_j in (16) and (17) show that each of the linear functions $1 \pm x_j$ is in Q. If

$$q = a + (1 - r)^2\, b, \qquad q_1 = a_1 + (1 - r^2)\, b_1$$

are in Q then

$$qq_1 = [aa_1 + (1 - r^2)^2\, bb_1] + (1 - r^2)[ab_1 + a_1 b]$$

is also in Q. Hence for any multi-indices α, β the product $(1 - x)^{\alpha}\, (1 + x)^{\beta}$ (see (15)) is also in Q.

Now suppose L satisfies conditions (i) and (ii). Then if a and b are finite sums of square of real polynomials,

$$L(a) \geqslant 0 \qquad \text{by} \quad \text{(i)},$$

$$L((1 - r^2)b) \geqslant 0 \qquad \text{by} \quad \text{(ii)}$$

so $L(q) \geqslant 0$ for every $q \in Q$. Hence, since $(1 - x)^{\alpha}\, (1 + x)^{\beta}$is in Q, L satisfies condition (15) of Theorem A. It follows that L is a C-moment functional.

Now choose any unit vector $\nu^{(1)} = (\nu_1^{(1)}, \ldots, \nu_N^{(1)})$ in E_N. The linear function

$$1 - (\nu^{(1)}, \xi) = 1 - (\nu_1^{(1)}\xi_1 + \cdots + \nu_N^{(1)}\xi_N)$$

defines a supporting half-space

$$H(\nu^{(1)}) = \{\xi;\ 1 - (\nu^{(1)}, \xi) \geqslant 0\}$$

for the unit sphere S, and all supporting half spaces are of this form. Choose additional unit vectors $\nu^{(2)}, \ldots, \nu^{(N)}$ which together with $\nu^{(1)}$ form an orthonormal basis in E_N. The identity

$$1 - (\nu^{(1)}, \xi) = \tfrac{1}{2}[\{1 - (\nu^{(1)}, \xi)^2 + (\nu^{(2)}, \xi)^2 + \cdots + (\nu^{(N)}, \xi)\}^2] + (1 - r^2) \cdot \tfrac{1}{2} \tag{18}$$

shows that $1 - (\nu^{(1)}, \xi) = f(\xi)$ is in Q, hence by (i) and (ii)

$$L(fpp^*) \geqslant 0 \qquad \text{for every} \quad p \text{ in } \mathscr{P}.$$

It then follows from Proposition 1 that the support of the representing measure μ of the C-moment functional L is contained in the half space $H(\nu^{(1)})$. Each unit vector ν defines a half space $H(\nu)$ which contains the support of μ. The intersection of all such half spaces is the unit sphere S so L is an S-moment functional and the proof is complete.

An alternative method for proving Theorem 1 is the operator theoretic method used in the proof of Theorem 6 below, with substantial simplifications because only bounded operators occur in the proof of Theorem 1. Alternatively Theorem 8.5 of [6] can be used.

Proof of Theorem 2

First, observe that the mapping T which sends a polynomial p into its restriction to Σ, denoted by $\hat{p}$, is an algebraic homomorphism of $\mathscr{P}$ onto $\mathscr{H}$. For each $p \in \mathscr{P}$ there is a unique harmonic polynomial h which is equal to p on Σ. Hence $\hat{p} = \hat{h}$ which is indeed a spherical harmonic, and clearly every spherical harmonic is in the range of T. Hence T maps $\mathscr{P}$ onto $\mathscr{H}$. The map is an algebraic homomorphism, that is T is linear, multiplicative $T(pq) = (Tp) \otimes Tq$, and respects the involution $T(p^*) = (Tp)^*$.

Now given a linear functional M on $\mathscr{H}$ which satisfies (iii), there is a linear functional L defined on $\mathscr{P}$ by

$$L(p) = M(Tp).$$

For any $p \in \mathscr{P}$

$$L(pp^*) = M(T(pp^*)) = M(T(p) \otimes (Tp)^*)$$

so $L(pp^*) \geqslant 0$ by (iii), i.e., L satisfies (i). Also for any $p \in \mathscr{P}$, since $T(1 - r^2) = 0$,

$$L((1 - r^2)\, pp^*) = M(T(1 - r^2)\, T(pp^*)) = 0.$$

Hence L satisfies (ii) and also the reversed inequality

$$\text{(ii)}' \qquad L((1 - r^2)\, pp^*) \leqslant 0 \qquad \text{for every} \quad p \in \mathscr{P}.$$

Because L satisfies (i) and (ii), L is an S-moment functional by Theorem 1. Because of (ii)$'$ the support of the representing measure μ is, by Proposition 1, contained in the set where $1 - r^2 \leqslant 0$. The intersection of this set with S is Σ, so the representing measure μ is supported on Σ.

The range of T is $\mathscr{H}$ so for each $\hat{h} \in \mathscr{H}$ there is a $p \in \mathscr{P}$ such that $Tp = \hat{h}$, and for any such p, $p = \hat{h}$ on Σ. Thus

$$M(\hat{h}) = M(Tp) = L(p) = \int_\Sigma p\, d\mu = \int_\Sigma \hat{h}\, d\mu.$$

Thus M is a Σ-moment functional. The additional clause in Theorem 2 concerning strict inequality in (iii) is evident.

Proof of Theorem 3

Let L satisfy conditions (i) and (iv) of Theorem 4. When both sides of the identity

$$2 - r^2 = (1 - s^2) + (1 - t^2) \tag{19}$$

are multiplied by pp^*, $p \in \mathscr{P}$, it is found from (iv) that L satisfies

$$L((2 - r^2)\, pp^*) \geqslant 0 \qquad \text{for every} \quad p \in \mathscr{P}. \tag{20}$$

By considering the effect on the conditions of Theorem 1 of a change of scale in E_N, it can be seen from (i) and (19) that L is an $S(\sqrt{2})$-moment functional where $S(\sqrt{2})$ is the sphere with center at the origin and radius $\sqrt{2}$. Since L also satisfies condition (iv), it follows from Proposition 1 that the representing measure μ for L has its support contained in the bi-sphere S^2, so L is an S^2-moment functional.

Proof of Theorem 4

Theorem 4 can be deduced from Theorem 3 in a way very similar to the way in which Theorem 2 was deduced from Theorem 1. The details are omitted.

Proof of Theorem 5

Let L satisfy conditions (i) and (vi). From the identity

$$4\tau(1 - \tau)(1 - r^2) = (1 - r^2)^2 + 4(1 - \tau)^2\, x_3^2 + f_T,$$

it follows that for every polynomial p

$$\begin{aligned} &4\tau(1 - \tau)\, L((1 - r^2)\, pp^*) \\ &\quad = L((1 - r^2)^2\, pp^*) + 4(1 - \tau)^2\, L(x_3^2\, pp^*) + L(f_T\, pp^*). \end{aligned}$$

The first two terms on the right side are $\geqslant 0$ by (i), the third is $\geqslant 0$ by (vi), and since $0 < 4\tau(1 - \tau)$ because $0 < \tau \leqslant \frac{1}{2}$, it follows that L satisfies condition (ii) of Theorem 1, so L is an S-moment functional. Then because of (vi), Proposition 1 ensures that the representing measure μ has support contained in $S \cap T = T$. Thus L is a T-moment functional.

Theorems 2, 3, 4, 5 have been obtained as simple deductions from Theorem 1. Many additional results of a similar nature can be obtained by similar arguments, for example, solvability criteria for hemispheres, spherical sectors, etc., and by changes of scale and other affine transformations in E_N, for ellipsoids and parts thereof.

The objective now is to prepare for the proof of Theorems 6 and 7. For the remainder of the discussion it is assumed that $\mathscr{P}$ is the algebra of all polynomials on E_{N+1}, and that L satisfies (i).

PROPOSITION 2. *Let L be a linear functional on $\mathscr{P}$ which satisfies* (i) *$L(pp^*) \geqslant 0$ for all $p \in \mathscr{P}$. Then $L(pq^*)$ is a semi-definite Hermitian form on $\mathscr{P}$, i.e., $L(pq^*)$ is linear in p, $L(pp^*) \geqslant 0$ and*

$$L(qp^*) = \overline{L(pq^*)}.$$

Proof. The last equality is obtained as follows. When p and q are interchanged in the polarization identity

$$\begin{aligned} 4L(pq^*) &= L((p+q)(p+q)^*) - L((p-q)(p-q)^*) \\ &\quad + i[L((p+iq)(p+iq)^*) - L((p-iq)(p-iq)^*)] \end{aligned}$$

the real part on the right is unchanged, the imaginary part changes sign.

The following deductions from the fact that $L(pq^*)$ is a semi-definite Hermitian form are well known (see [6, Chap. 8]). The form satisfies the Schwarz inequality

$$|L(pq^*)| \leqslant L^{1/2}(pp^*)\, L^{1/2}(qq^*)$$

and from this it is seen that the set

$$\begin{aligned} \mathscr{Q} &= \{q; L(qq^*) = 0\} \\ &= \{q; L(pq^*) = 0 \text{ for all } p \in \mathscr{P}\} \end{aligned}$$

is a self adjoint ideal in $\mathscr{P}$. The quotient algebra $\mathscr{P}/\mathscr{Q}$ is a complex algebra with an identity element $1 + \mathscr{Q}$ and an involution $p + \mathscr{Q} \to p^* + \mathscr{Q}$. The formula

$$(p_1 + \mathscr{Q}, p_2 + \mathscr{Q}) = L(p_1 p_2^*)$$

defines an inner product in $\mathscr{P}/\mathscr{Q}$ with corresponding norm

$$\|p + \mathscr{Q}\| = L^{1/2}(pp^*). \tag{21}$$

The completion of $\mathscr{P}/\mathscr{Q}$ relative to this norm is a Hilbert space $\mathscr{H}$ containing $\mathscr{P}/\mathscr{Q}$ as a dense subspace. Elements of $\mathscr{H}$ will be denoted by $a, b, c, \ldots$ sometimes with subscripts.

The involution of $\mathscr{P}/\mathscr{Q}$ is a conjugate-linear map of $\mathscr{P}/\mathscr{Q}$ onto itself, isometric in the norm of $\mathscr{H}$, so has a unique continuous extension to a continuous map J of $\mathscr{H}$ onto $\mathscr{H}$. The extended map J is a conjugation of $\mathscr{H}$, that is, J is conjugate linear, $J^2 = I$, and for all a, b in $\mathscr{H}$

$$(Ja, Jb) = (b, a).$$

For $1 \leqslant k \leqslant N+1$, multiplicationn by $x_k + \mathscr{Z}$ in the algebra $\mathscr{P}/\mathscr{Z}$ can be viewed as a linear map X_k of $\mathscr{H}$ with domain $\mathscr{P}/\mathscr{Z}$. Te operators X_k are commuting symmetric operators which map their common dense domain into itself, and they commute with the conjugation J,

$$X_k J = J X_k, \qquad 1 \leqslant k \leqslant N+1.$$

PROPOSITION 3. *If L satisfies conditions* (i) *and* (vii) *then the operators X_k, $k = 1,..., N$, have unique extensions to bounded self adjoint operators A_k which commute with each other, with J, and satisfy*

$$\sum_{k=1}^{N} (A_k^2 a, a) \leqslant (a, a) \qquad \textit{for all} \quad a \in \mathscr{H}. \tag{22}$$

Let $\mathscr{A}$ denote the C^*-algebra generated by the identity and $A_1,..., A_N$, and let S denote the closure of X_{N+1}. Then is commutative, and every T in $\mathscr{A}$ satisfies $JT = T^*J$. Moreover $SJ = JS$ and every T in $\mathscr{A}$ commutes with S, that is, $ST \supseteq TS$.

Proof. For any $p \in \mathscr{P}$ the identity

$$\sum_{k=1}^{N} \|X_k(p + \mathscr{Z})\|^2 = \sum_{k=1}^{N} L(x_k^2\, pp^*) = L(r_N^2\, pp^*) \tag{23}$$

and the inequality resulting from (vii)

$$L(r_N^2\, pp^*) \leqslant L(pp^*) = \|p + \mathscr{Z}\|^2 \tag{24}$$

show that each X_k, $k = 1,..., N$, is bounded, so has a unique extension to an everywhere defined bounded operator A_k. That the A_k are self adjoint and commute with each other and with J follows by continuity from the symmetry and the corresponding commuting properties of the X_k. The inequality (22) follows by continuity from (23) and (24).

If f is a polynomial in N variables with complex coefficients, there is a corresponding operator polynomial $f(A_1,..., A_N)$ which will be denoted by $f(A)$. Every element of $\mathscr{A}$ is the limit in the uniform operator topology, of a sequence of operator polynomials.

Let $T \in \mathscr{A}$ and $f_m(A)$ be operator polynomials such that $f_m(A) \to T$. Then for any $p \in \mathscr{P}$

$$f_m(A)(p + \mathscr{Z}) \to T(p + \mathscr{Z}),$$

and

$$\begin{aligned} X_{N+1} f_m(A)(p+\mathscr{Z}) &= x_{N+1} f_m(x_1, \ldots, x_N)\, p + \mathscr{Z} \\ &= f_m(A)\, x_{N+1}\, p + \mathscr{Z} \\ &= f_m(A)\, S(p+\mathscr{Z}) \\ &\to TS(p+\mathscr{Z}). \end{aligned}$$

which shows that $T(p+\mathscr{Z}) \in \mathscr{D}(S)$ and

$$ST(p+\mathscr{Z}) = TS(p+\mathscr{Z}).$$

Now suppose $a \in \mathscr{D}(S)$. Then there is a sequence $p_n \in \mathscr{P}$ such that

$$p_n + \mathscr{Z} \to a, \qquad S(p_n + \mathscr{Z}) \to Sa.$$

Since T is bounded and $T(p_n + \mathscr{Z}) \in \mathscr{D}(S)$, $T(p_n + \mathscr{Z}) \to Ta$ and

$$ST(p_n + \mathscr{Z}) = TS(p_n + \mathscr{Z}) \to TSa.$$

Hence $Ta \in \mathscr{D}(S)$ and $STa = TSa$, thus $ST \supseteq TS$.

With the same sequence p_n,

$$p_n^* + \mathscr{Z} = J(p_n + \mathscr{Z}) \to Ja,$$

$$SJ(p_n + \mathscr{Z}) = x_{N+1}\, p_n^* + \mathscr{Z} = JS(p_n + \mathscr{Z}) \to JSa$$

so $Ja \in \mathscr{D}(S)$ and $SJa = JSa$. Since $J^2 = I$ it is clear that $SJ = JS$. Q.E.D.

PROPOSITION 4. *Let $\mathscr{H}$ be a Hilbert space with a conjugation J and let $\mathscr{A}$ be a C^* algebra acting on $\mathscr{H}$, containing the identity and such that $JT = T^*J$ for $T \in \mathscr{A}$. A densely defined symmetric operator S will be said to have the commutation properties if*

$$JS = SJ \qquad \textit{and} \qquad ST \supseteq TS \qquad \textit{for} \quad T \in \mathscr{A}. \tag{25}$$

I. *Any densely defined closed symmetric operator S with the commutation properties has a self adjoint extension with the commutation properties.*

II. *If also S is positive, i.e., $(Sa, a) \geqslant 0$ for $a \in \mathscr{D}(S)$ then S has a positive definite self adjoint extension with the commutation properties.*

Remark. Phillips [9, footnote p. 382] has given an example of a densely defined symmetric operator with equal deficiency indices commuting elementwise with a one parameter unitary group but lacking any self adjoint extension so commuting. This example has no counterpart of the conjugation J.

Proof. The proof uses the theory of Cayley transforms and the Friedrichs extension, both found in [5, Chap. 8]. If S is self adjoint the two conclusions are true. Assume S is not self adjoint.

The Cayley transform V of S is the closed linear isometric operator $V = (S - i)(S + i)^{-1}$ with domain and range the closed proper subspaces

$$\mathscr{D}(V) = \mathscr{R}(S + i), \qquad \mathscr{R}(V) = \mathscr{R}(S - i).$$

Since J maps $\mathscr{D}(S)$ onto itself it is easily seen that J maps $\mathscr{D}(V)$ and $\mathscr{R}(V)$ each onto the other, and from the commutation property of S, $JV = V^{-1}J$. Again using the commutation, any $T \in a$ maps $\mathscr{D}(V)$ and $\mathscr{R}(V)$ each into itself and if $c \in \mathscr{D}(V)$, $c_1 \in \mathscr{R}(V)$ then

$$VTc = TVc, \qquad V^{-1}Tc_1 = TV^{-1}c_1.$$

The defect spaces of S are the orthogonal complements

$$\mathscr{D}(+) = \mathscr{D}(V)^{\perp}, \qquad \mathscr{D}(-) = \mathscr{R}(V)^{\perp}.$$

From $(Jc, Jb) = (b, c)$ applied with $c \in \mathscr{D}(V)$ or $\mathscr{R}(V)$ and b in the complement it follows that J maps the spaces $\mathscr{D}(+)$, $\mathscr{D}(-)$ each onto the other. By similar consideration of $(c, Tb) = (T^*c, b)$, $T \in \mathscr{A}$, it follows since $\mathscr{A}$ is self adjoint that T maps the spaces $\mathscr{D}(+)$, $\mathscr{D}(-)$ each into itself.

If V_1 is a unitary extension of V then $I - V_1$ has an inverse and

$$S_1 = i(I + V_1)(I - V_1)^{-1}$$

is a self adjoint extension of S. This formula establishes a 1–1 correspondence between all self adjoint extensions S_1 of S and all unitary extensions V_1 of V. In order that S_1 have the commutation properties it is necessary and sufficient that

$$JV_1 = V_1^{-1}J, \qquad V_1 T = TV_1 \qquad \text{for} \quad T \in \mathscr{A}. \tag{26}$$

Any $h \in \mathscr{H}$ has a unique representation as an orthogonal sum

$$h = c + z, \qquad c \in \mathscr{D}(V), \qquad z \in \mathscr{D}(+). \tag{27}$$

If V_1 is a unitary extension of V then $V_1 h = Vc + V_1 z$ and $z \to V_1 z$ is a linear isometric map of $\mathscr{D}(+)$ onto $\mathscr{D}(-)$. Conversely, if W_1 is any linear isometric map of $\mathscr{D}(+)$ onto $\mathscr{D}(-)$ the formula

$$V_1 h = Vc + W_1 z \tag{28}$$

defines a unitary extension V_1 of V. In order that V_1 satisfy (26) it is necessary and sufficient that

$$JW_1 = W_1^{-1}J, \qquad W_1 Tz = TW_1 z \qquad \text{for} \quad T \in \mathscr{A}, \quad z \in \mathscr{L}(+). \tag{29}$$

The restrictions of the $T \in \mathscr{A}$ to $\mathscr{L}(+)$ form a C^*-algebra on $\mathscr{L}(+)$ which is commutative and contains the identity. Hence [6, Chap. 9], $\mathscr{L}(+)$ is the orthogonal sum of a family of cyclic subspaces $\{\mathscr{L}(+, \nu)\}$ with cyclic vectors $\{z_\nu\}$. It follows that $\mathscr{L}(-)$ is the orthogonal sum of the cyclic subspaces $\mathscr{L}(-, \nu) = J\mathscr{L}(+, \nu)$ with cyclic vectors Jz_ν. The collection of all finite sums

$$z = \Sigma T_\nu z_\nu, \qquad T_\nu \in \mathscr{A} \tag{30}$$

is a dense subspace of $\mathscr{L}(+)$. If the vector z in (30) is $z = 0$ then, from orthogonality, $T_\nu z_\nu = 0$ for each ν, and since $\mathscr{A}$ is commutative, $T_\nu u = 0$ for all $u \in \mathscr{L}(+, \nu)$. In this sense the representation (30) is unique, and in terms of it

$$W_0 z = \Sigma T_\nu J z_\nu \tag{31}$$

defines an isometric linear map of a dense subspace of $\mathscr{L}(+)$ onto a dense subspace of $\mathscr{L}(-)$. Let W_1 be the extension of W_0 to an isometric linear map of $\mathscr{L}(+)$ onto $\mathscr{L}(-)$. Simple calculations show that W_0, and by continuity also W_1, satisfies (29). With this W_1, (28) defines a unitary extension of V satisfying (26), and I is proved.

The proof of part II, by showing that the Friedrichs extension has the commutation properties, is given in [9], except for minor changes to accomodate the conjugation.

The next proposition can be used to obtain numerous interesting consequences from Theorems 6, 7 in the same way that Proposition 1 was used to obtain Theorems 2, 3, 4, 5 as corollaries of Theorem 1. The proposition studies cylinders in E_{N+M} and their forebearing sets in E_N. The forebearance is determined by the projection map τ of E_{N+M} onto E_N defined by

$$\tau((\xi_1, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{N+M})) = (\xi_1, \ldots, \xi_N)$$

and its inverse.

PROPOSITION 5. *Let K be a compact set in E_N and let L be an E_{N+M} moment functional with a representing measure μ (not necessairly unique) whose support lies in the cylinder $\tau^{-1}(K)$. If f is a polynomial on E_N such that for every polynomial p on E_N*

$$L(fpp^*) \geqslant 0$$

then the support of μ lies in the cylinder $\tau^{-1}(K(f))$, where

$$K(f) = \{\xi; \xi \in K, f(\xi) \geqslant 0\}.$$

Proof. For any Borel set H in E_N the formula

$$\nu(H) = \mu(\tau^{-1}(H))$$

defines a non-negative Borel measure ν on E_N with support in K. If p is any polynomial on E_N then

$$\int_{E_N} p\, d\nu = \int_{E_{N+M}} p\, d\mu = L(p),$$

so the restriction L_N of L to polynomials in the first N variables is an E_N moment functional with representing measure ν. Since $L_N(fpp^*) \geqslant 0$ it follows from Proposition 1 that the support of ν lies in $K(f)$, and hence the support of μ in $\tau^{-1}(K(f)$.

Proof of Theorem 6

With the notation of Proposition 3 let A_{N+1} be a self adjoint extension of S with the commutation properties of Proposition 4 relative to J and the C^* algebra generated by $A_1, \ldots, A_N$ and I. Let $E_k(\xi)$ be the resolution of the identity of for A_k, $k = 1, \ldots, N+1$. Because of the commutative properties of the A_k the formula

$$E(\xi) = E_1(\xi_1) \cdots E_{N+1}(\xi_{N+1}), \qquad \xi \in E_{N+1}$$

defines a resolution of the identity on E_{N+1}. The function

$$\mu(\xi_1, \ldots, \xi_{N+1}) = (E_1(\xi_1) \cdots E_{N+1}(\xi_{N+1})(1 + \mathscr{Z}), 1 + \mathscr{Z})$$

is increasing in each of the $N+1$ variables and so determines a non-negative measure, also denoted by μ, on E_{N+1}. If p is any polynomial

$$\begin{aligned} L(p) &= (p + \mathscr{Z}, 1 + \mathscr{Z}) \\ &= (p(A_1, \ldots, A_{N+1})(1 + \mathscr{Z}), (1 + \mathscr{Z})) \\ &= \int_{E_{N+1}} p\, d\mu, \end{aligned}$$

so L is an E_{N+1} moment functional with representing measure μ. From $\|A_k\| \leqslant 1$, $k = 1, \ldots, N$, it follows that for $k \leqslant N$

$$\begin{aligned} E_k(\xi_k) &= 0, \qquad \xi_k < -1 \\ &= I, \qquad \xi_k > +1 \end{aligned}$$

and thus the support of μ is contained in the cylinder $\tau^{-1}(C)$, where C is the unit cube of E_N as in (13). Then since $L((1-r_N^2)\,pp^*)\geqslant 0$, from Proposition 5 with $f=1-r_N^2$, the support of μ is contained in the spherical cylinder (Eq. (11)),

$$\tau^{-1}(C(1-r_N^2))=Z, \qquad \text{Q.E.D.}$$

Proof of Theorem 7

Let A_{N+1} be a positive definite self adjoint extension of S with the commutation properties. Then the conclusions of Theorem 6 are valid, and from the positivity of A_{N+1} follows $E_{N+1}(\xi_{N+1})=0$ for $\xi_{N+1}<0$, so in this case the support of μ is contained in the half-space where $\xi_{N+1}\geqslant 0$.

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